

DETERMINANTS¹

Cyclic permutations. A *permutation* is a one-to-one mapping of a set onto itself. A *cyclic permutation*, or a *cycle*, or a *k-cycle*, where $k \geq 2$ is an integer, is a permutation σ where for some elements i_1, i_2, \dots, i_k , we have $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \dots, \sigma(i_{k-1}) = i_k, \sigma(i_k) = i_1$. A standard notation for this permutation is $\sigma = (i_1 i_2 \dots i_k)$. One often considers this σ as being a permutation of some set M that includes the set $\{i_1, i_2, \dots, i_k\}$ such that $\sigma(i) = i$ for any element of M that is not in this set. The two-cycle (ii) is the identity mapping. Such a two-cycle is not a *proper* two-cycle, a proper two-cycle being a two-cycle (ij) for i and j distinct. A proper two-cycle is also called a *transposition*.

Lemma. *Every permutation of a finite set can be written as a product of transpositions.*

Proof. It is enough to consider permutations of the set $M_n \stackrel{\text{def}}{=} \{1, 2, \dots, n\}$. We will use induction on n . The Lemma is certainly true for $n = 1$, in which case every permutation of M_n (the identity mapping being the only permutation) can be written as a product of zero number of transpositions (the *empty* product of mappings will be considered to be the identity mapping). Let σ be a permutation of the set M_n , and assume $\sigma(n) = i$. Then

$$(in)\sigma(n) = n$$

(since $i = \sigma(n)$ and $(in)(i) = n$). So the permutation $\rho = (in)\sigma$ restricted to the set M_{n-1} is a permutation of this latter set. By induction, it can be written as a product $\tau_1 \dots \tau_m$ of transpositions, and so

$$\sigma = (in)^{-1}\rho = (in)\rho = (in)\tau_1 \dots \tau_m,$$

where for the second equality, one needs to note that, clearly, the inverse $(in)^{-1}$ of (in) is (in) itself. \square

One can use the idea of the proof of the above Lemma, or else one can use direct calculation, to show that if i, j , and k are distinct elements then

$$(ijk) = (ik)(ij).$$

For example, $(ij)(j) = i$ and $(ik)(i) = k$, and so $(ik)(ij)(j) = k$, which agrees with the equation $(ijk)(j) = k$.

Even and odd permutations. We start with the following

Lemma. *The identity mapping cannot be written as a product of an odd number of transpositions.*

Proof. We will assume that the underlying set of the permutations is M_n , and we will use induction on n . For $n = 1$ the statement is certainly true; namely, there are no transpositions on M_1 , so the identity mapping can be represented only as a product of a zero number of transpositions. According to the last displayed equation, if i, j , and n are distinct elements of M_n , we have $(nij) = (nj)(ni)$, and we also have $(nij) = (ijn) = (in)(ij) = (ni)(ij)$, and so

$$(nj)(ni) = (ni)(ij).$$

Moreover, in a similar way, $(nij) = (jni) = (ji)(jn) = (ij)(nj)$. Comparing this with one of the expressions equal to (nij) above we obtain

$$(ij)(nj) = (ni)(ij).$$

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Further, we clearly have

$$(ni)(ni) = \iota,$$

where ι (the Greek letter *iota*) stands for the identity mapping (often represented by the empty product). Finally, if i, j, k, n are distinct, then we have

$$(ij)(nk) = (nk)(ij).$$

Using these four displayed equations, all transpositions containing n in the product can be moved all the way to the left in such a way that in the end either no transposition will contain n , or at most one transposition containing n will remain all the way to the left. Each application of these identities changes the number of transpositions in the product by an even number; in fact, the third of these identities decreases the number of transpositions by two, and the others do not change the number of transpositions.

A product $\sigma = (ni)\tau_1 \dots \tau_m$, where i is distinct from n , and the transpositions τ_1, \dots, τ_m do not contain n cannot be the identity mapping (since $\sigma(n) = i$), so the only possibility that remains is that we end up with a product of transpositions representing the identity mapping where none of the transpositions contain n . Then we can remove n from the underlying set; since, by the induction hypothesis, the identity mapping on M_{n-1} can only be represented as a product of an even number of transpositions, we must have started with an even number of transpositions to begin with.

Corollary. *A permutation cannot be written as a product both of an even number of transpositions and of an odd number of transpositions.*

Proof. Assume that for a permutation σ we have $\sigma = \tau_1 \dots \tau_k$ and $\sigma = \rho_1 \dots \rho_l$ where k is even, l is odd, and τ_1, \dots, τ_k , and ρ_1, \dots, ρ_l , are transpositions. The identity mapping can be written as a product

$$\iota = \sigma\sigma^{-1} = \tau_1 \dots \tau_k (\rho_1 \dots \rho_l)^{-1} = \tau_1 \dots \tau_k (\rho_l)^{-1} \dots (\rho_1)^{-1} = \tau_1 \dots \tau_k \rho_l \dots \rho_1.$$

Since the right-hand side contains an odd number of transpositions, this is impossible according to the last Lemma. \square

A permutation that can be written as a product of an even number of transpositions is called an *even permutation*, and a permutation that can be written as a product of an odd number of transpositions is called an *odd permutation*. The function sgn (*sign*, or Latin *signum*) on permutations of a finite set is defined as $\text{sgn}(\sigma) = 1$ if σ is an even permutation, and $\text{sgn}(\sigma) = -1$ if σ is an odd permutation. Often, it is expedient to extend the signum function to mappings of a finite set into itself that are not permutations (i.e., that are not one-to-one) by putting $\text{sgn}(\sigma) = 0$ if σ is not one-to-one.

The distributive rule. Let (a_{ij}) be an $n \times n$ matrix. Then

$$\prod_{i=1}^n \sum_{j=1}^n a_{ij} = \sum_{\sigma} \prod_{i=1}^n a_{i\sigma(i)},$$

where σ runs over all mappings (not necessarily one-to-one) of the set $M_n = \{1, 2, \dots, n\}$ into itself. The left-hand side represents a product of sums. The right-hand side multiplies out this product by taking one term out of each of these sums, and adding up all the products that can be so formed. The equality of these two sides is obtained by the distributivity of multiplication over addition. For example, for $n = 2$, the above equation says that

$$(a_{11} + a_{12})(a_{21} + a_{22}) = a_{1\sigma_1(1)}a_{2\sigma_1(2)} + a_{1\sigma_2(1)}a_{2\sigma_2(2)} + a_{1\sigma_3(1)}a_{2\sigma_3(2)} + a_{1\sigma_4(1)}a_{2\sigma_4(2)},$$

where $\sigma_1(1) = 1, \sigma_1(2) = 1, \sigma_2(1) = 1, \sigma_2(2) = 2, \sigma_3(1) = 2, \sigma_3(2) = 1, \sigma_4(1) = 2, \sigma_4(2) = 2$.

Determinants. The determinant $\det A$ of an $n \times n$ matrix $A = (a_{ij})$ is defined as

$$\det A = \sum_{\sigma} \text{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)},$$

where σ runs over all mappings (not necessarily one-to-one) of the set $M_n = \{1, 2, \dots, n\}$ into itself. The above formula is called the *Leibniz formula* for determinants. Sometimes one writes $\det(A)$ instead of $\det A$. Since $\text{sgn}(\sigma) = 0$ unless σ is a permutation, one may say instead that σ runs through all permutations of the set of the set $M_n = \{1, 2, \dots, n\}$. However, for some considerations (e.g., for the application of the distributive rule above) it may be expedient to say that σ runs through all mappings, rather than just permutations.

Multiplications of determinants. The product of two determinants of the same size is the determinant of the product matrix; that is:

Lemma. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices. Then*

$$\det(AB) = \det(A) \det(B).$$

Proof. With σ and ρ running over all permutations of M_n , we have

$$\begin{aligned} \det(A) \det(B) &= \sum_{\sigma, \rho} \operatorname{sgn}(\sigma) \operatorname{sgn}(\rho) \left(\prod_{i=1}^n a_{i\sigma(i)} \right) \left(\prod_{i=1}^n b_{i\rho(i)} \right) \\ &= \sum_{\sigma, \rho} \operatorname{sgn}(\rho) \operatorname{sgn}(\sigma) \left(\prod_{i=1}^n a_{i\sigma(i)} \right) \left(\prod_{i=1}^n b_{\sigma(i)\rho\sigma(i)} \right) = \sum_{\sigma, \rho} \operatorname{sgn}(\rho\sigma) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\rho\sigma(i)} \end{aligned}$$

On the right-hand side of the second equality, the product $\prod_{i=1}^n b_{\sigma(i)\rho\sigma(i)}$ is just a rearrangement of the product $\prod_{i=1}^n b_{i\rho(i)}$, since σ is a one-to-one mapping. The third equality takes into account that $\operatorname{sgn}(\rho) \operatorname{sgn}(\sigma) = \operatorname{sgn}(\rho\sigma)$ (since, if ρ can be written as the product of k transpositions and σ as the product of l transpositions, $\rho\sigma$ can be written as the product of $k+l$ transpositions). Writing $\rho\sigma = \pi$, the permutations π and σ uniquely determine ρ , so the right-hand side above can be written as

$$\sum_{\sigma, \pi} \operatorname{sgn}(\pi) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(i)} = \sum_{\sigma} \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(i)}.$$

Here σ and π run over all permutations of M_n . Now we want to change our point of view in that we want to allow σ to run over all mappings of M_n into itself on the right-hand side, while we still restrict π to permutations.² To show that this is possible, we will show that the inner sum is zero whenever σ is not a permutation. In fact, assume that for some distinct k and l in M_n we have $\sigma(k) = \sigma(l)$. Then, denoting by (kl) the transposition of k and l , the permutation $\pi(kl)$ will run over all permutations of M_n as π runs over all permutations of M_n . Hence, the first equality next is obvious:

$$\begin{aligned} \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(i)} &= \sum_{\pi} \operatorname{sgn}(\pi(kl)) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(kl)(i)} \\ &= - \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(kl)(i)} = - \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(i)}. \end{aligned}$$

The second equality expresses the fact that $\operatorname{sgn}(\pi(kl)) = -\operatorname{sgn}(\pi)$, and the third equality expresses the fact that $\sigma(k) = \sigma(l)$, and the equation just reflects the interchange of the factors corresponding to $i = k$ and $i = l$ in the product. Rearranging this equation, it follows that

$$2 \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(i)} = 0,$$

and so

$$\sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(i)} = 0,$$

as we wanted to show.³

²One might ask why did we not extend the range of σ to all mappings earlier, when this would have been easy since we had $\operatorname{sgn}(\sigma)$ in the expression, which is zero if σ is not one-to-one. The answer is that we wanted to make sure that $\pi = \rho\sigma$ is a permutation, and if σ is not one-to-one then $\pi = \rho\sigma$ is not one-to-one, either.

³The theory of determinants can be developed for arbitrary rings. For rings of characteristic 2 (in which one can have $a + a = 0$ while $a \neq 0$ – in fact, $a + a = 0$ holds for every a), the last step in the argument is not correct. It is, however, easy to change the argument in a way that it will also work for rings of characteristic 2. To this end, one needs to split up the summation for π into two parts such that in the first part π runs over all even permutations; then $\pi(kl)$ will run over all odd permutations, and then one needs to show that these two parts of the sum cancel each other.

Therefore, in the last sum expressing $\det(A) \det(B)$ one can allow σ to run over all mappings of M_n into itself, and not just over permutations (while π will still run over all permutations). We have

$$\begin{aligned} \det(A) \det(B) &= \sum_{\sigma} \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(i)} = \sum_{\pi} \operatorname{sgn}(\pi) \sum_{\sigma} \prod_{i=1}^n a_{i\sigma(i)} b_{\sigma(i)\pi(i)} \\ &= \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^n \sum_{k=1}^n a_{ik} b_{k\pi(i)} = \det \left(\sum_{k=1}^n a_{ik} b_{kj} \right)_{i,j} = \det(AB); \end{aligned}$$

the third equality follows by the distributive rule mentioned above, and the last equality holds in view of the definition of the product matrix AB . \square

Simple properties of determinants. For a matrix A , $\det A = \det A^T$, where A^T denotes the transpose of A . Indeed, if $A = (a_{ij})$ then, with σ running over all permutations of M_n , we have

$$\begin{aligned} \det A^T &= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(i)i} = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma^{-1}(i)} = \sum_{\sigma} \operatorname{sgn}(\sigma^{-1}) \prod_{i=1}^n a_{i\sigma^{-1}(i)} \\ &= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)} = \det A; \end{aligned}$$

here the second equality represents a change in the order in of the factors in the product, the third equality is based on the equality $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$, and the fourth equality is obtained by replacing σ^{-1} with σ , since σ^{-1} runs over all permutations of M_n , just as σ does.

If one interchanges two columns of a determinant, the value determinant gets multiplied by -1 . Formally, if $k, l \in M_n$ are distinct, then $\det(a_{ij})_{i,j} = -\det(a_{i(kl)(j)})_{i,j}$, where, as usual, (kj) denotes a transposition. Indeed, with σ running over all permutations of M_n , we have

$$\begin{aligned} \det(a_{i(kl)(j)})_{i,j} &= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(kl)(i)} = -\sum_{\sigma} \operatorname{sgn}(\sigma(kl)) \prod_{i=1}^n a_{i\sigma(kl)(i)} \\ &= -\sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)} = -\det A; \end{aligned}$$

the second equality holds, since $\operatorname{sgn}(\sigma(kl)) = -\operatorname{sgn}(\sigma)$, and the third equality is obtained by replacing $\sigma(kl)$ by σ , since $\sigma(kl)$ runs over all permutations of M_n , just as σ does. Of course, since $\det A^T = \det A$, a similar statement can be made when one interchanges rows.

If two columns of A are identical, then $\det A = 0$. Indeed, by interchanging the identical columns, one can conclude that $\det A = -\det A$.⁴

If we multiply a row of a determinant by a number c , then the determinant gets multiplied by c . Formally: if $A = (a_{ij})$ $B = (b_{ij})$, and for some $k \in M_n$ and some number c , we have $b_{ij} = a_{ij}$ if $i \neq k$ and we have $b_{kj} = ca_{kj}$ then $\det B = c \det A$. This is easy to verify by factoring out c from each of the products in the defining equation of the determinant $\det B$.

If two determinants are identical except for one row, then the determinant formed by adding the elements in the different rows, while keeping the rest of the elements unchanged, the two determinants get added. Formally, if $A = (a_{ij})$ $B = (b_{ij})$, and for some $k \in M_n$ we have $b_{ij} = a_{ij}$ if $i \neq k$, and $C = (c_{ij})$, where we have $c_{ij} = a_{ij}$ if $i \neq k$, and $c_{kj} = a_{kj} + b_{kj}$, then $\det C = \det A + \det B$. This is again easy to verify from the defining equation of determinants.

If one adds the multiple of a row to another row in a determinant, then the value of the determinant does not change. To see this, note that adding c times row l to row k to a determinant amounts to adding to the determinant c times a second determinant in which rows k and l are identical; since this second determinant is zero, nothing will change.

⁴This argument does not work for rings of characteristic 2. In order to establish the result for this case as well, one needs to split up the sum representing the determinant into sums containing even and odd permutations, respectively, as pointed out in the previous footnote.

Expansion of a determinant by a row. Given an $n \times n$ matrix $A = (a_{ij})$, denote by $A(i, j)$ the $(n - 1) \times (n - 1)$ matrix obtained by deleting the i th row and j th column of the matrix.

Lemma. *If a_{11} is the only (possibly) nonzero element of the first row of A , then*

$$\det A = a_{11} \det A(1, 1).$$

By “(possibly) nonzero” we mean that the Lemma of course applies also in the case when we even have $a_{11} = 0$.

Proof. With σ running over all permutations of M_n and ρ running over all permutations of $\{2, \dots, n\}$, we have

$$\begin{aligned} \det A &= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)} = \sum_{\sigma: \sigma(1)=1} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \\ &= \sum_{\sigma: \sigma(1)=1} a_{11} \operatorname{sgn}(\sigma) \prod_{i=2}^n a_{i\sigma(i)} = a_{11} \sum_{\rho} \operatorname{sgn}(\rho) \prod_{i=2}^n a_{i\rho(i)} = a_{11} \det A(1, 1); \end{aligned}$$

for the second equality, note that the product on the left-hand side of the second equality is zero unless $\sigma(1) = 1$. For the fourth equality, note that if ρ is the restriction to the set $\{2, \dots, n\}$ of a permutation σ of M_n with $\sigma(1) = 1$, then $\operatorname{sgn}(\rho) = \operatorname{sgn}(\sigma)$.

Corollary. *If for some $k, l \in M_n$, a_{kl} is the only (possibly) nonzero element in the k th row of A , then $\det A = (-1)^{k+l} a_{kl} \det A(k, l)$.*

Proof. The result can be obtained from the last Lemma by moving the element a_{kl} into the top left corner (i.e., into position $(1, 1)$) of the matrix A . However, when doing this, it will not work to interchange the k th row of the matrix A with the first row, since this will change the order of rows in the submatrix corresponding to the element. In order not to disturb the order of rows in the submatrix $A(k, l)$, one always needs to interchange adjacent rows. Thus, one can move the k th row into the position of the first row by first interchanging rows k and $k - 1$, then rows $k - 1$ and $k - 2$, then rows $k - 2$ and $k - 3$, etc. After bringing the element a_{kl} into the first row, one can make similar column interchanges. While doing so, one makes altogether $k - 1 + l - 1$ row and column interchanges, hence the factor $(-1)^{k+l} = (-1)^{(k-1)+(l-1)}$ in the formula to be proved.

The following theorem describes the expansion of a determinant by a row. It is usually attributed to Pierre-Simon Laplace (1749–1827), but it was known to Gottfried Wilhelm Leibniz (1646–1716), who invented determinants of order greater than two.

Theorem. *For any integer $k \in M_n$ we have*

$$\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A(i, j).$$

Proof. Let the matrix B_j be the matrix that agrees with A except for row k , and in row k all elements are zero except that the element in position (k, j) is a_{kj} . In view of the last Corollary, the equation to be proved can be written as

$$\det A = \sum_{j=1}^n \det B_j;$$

this equation can be established by the (repeated) use of the addition rule of determinants.

The number $A_{ij} \stackrel{\text{def}}{=} (-1)^{i+j} \det A(i, j)$ is often called the *cofactor* of the entry a_{ij} of the matrix A ; then the above equation can be written as

$$\det A = \sum_{j=1}^n a_{ij} A_{ij}.$$